

# Analytic Continuation of Bernoulli Numbers, a New Formula for the Riemann Zeta Function, and the Phenomenon of Scattering of Zeros <sup>1</sup>

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## Abstract

The method analytic continuation of operators acting integer  $n$ -times to complex  $s$ -times (hep-th/9707206) is applied to an operator that generates Bernoulli numbers  $B_n$  (Math. Mag., **70**(1), 51 (1997)).  $B_n$  and Bernoulli polynomials  $B_n(s)$  are analytic continued to  $B(s)$  and  $B_s(z)$ . A new formula for the Riemann zeta function  $\zeta(s)$  in terms of nested series of  $\zeta(n)$  is derived. The new concept of dynamics of the zeros of analytic continued polynomials is introduced, and an interesting phenomenon of ‘scatterings’ of the zeros of  $B_s(z)$  is observed.

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# 1 Introduction: Bernoulli Numbers

Bernoulli numbers  $B_n$  were discovered by Jakob Bernoulli (1654-1705) [1]. They are defined [2] [3] as

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \quad |z| < 2\pi, \quad n = 1, 2, 3, \dots \in \mathbb{Z}^+ \quad (1)$$

Expanding the l.h.s. as a series and matching the coefficients on both sides gives

$$B_1 = -1/2, \quad B_n \begin{cases} = 0 & , \text{ odd } n \\ \neq 0 & , \text{ even } n \end{cases} \quad (2)$$

With this result, (1) can be rewritten as

$$\frac{z}{e^z - 1} + \frac{z}{2} = \sum_{n=0}^{\infty} \frac{B_{2n}}{2n!} z^{2n} \quad (3)$$

Alternatively, Bernoulli numbers can be defined as satisfying the recurrence relation

$$B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k, \quad B_0 = 1 \quad (4)$$

Bernoulli numbers are interesting numbers. They appear in connection with a wide variety of areas, from Euler-Maclaurin Summation formula in Analysis [4] [5] and the Riemann zeta function in Number Theory [6] [7], to Kummer's regular primes in special cases of Fermat's Last Theorem and Combinatorics [8].

# 2 A Tree for Generating Bernoulli Numbers

It was shown in [4] how a binary Tree for generating Bernoulli numbers can be intuited step-by-step and eventually discovered. In the process of calculating the analytic continuation of the Riemann zeta function to the negative half plane term-by-term, an emerging pattern was observed. The big picture of the structure of the Tree became apparent on comparing the derived expressions with the Euler-Maclaurin Summation formula.

In this paper, we start with the Tree and proceed on to find interesting applications. While doing so, we will encounter some surprising consequences.

The Tree can be constructed using two operators,  $O_L$  and  $O_R$ .

At each node of the Tree sits a formal expression of the form  $\frac{\pm 1}{a! b! \dots}$ .

Define  $O_L$  and  $O_R$  to act only on formal expressions of this form at the nodes of the Tree as follows:

$$O_L : \frac{\pm 1}{a! b! \dots} \rightarrow \frac{\mp 1}{(a+1)! b! \dots} \quad (5)$$

$$O_R : \frac{\pm 1}{a! b! \dots} \rightarrow \frac{\pm 1}{2! a! b! \dots} \quad (6)$$

Schematically,

- $O_L$  acting on a node of the Tree generates a branch downwards to the left (hence the subscript  $_L$  in  $O_L$ ) with a new node at the end of the branch.

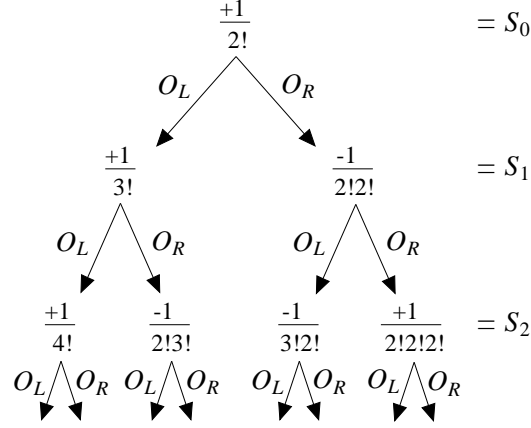


Figure 1: The binary Tree that generates Bernoulli numbers.

- $O_R$  acting on the same node generates a branch downwards to the right.

Form a finite series out of the sum of the two non-commuting operators

$$S_n = (O_L + O_R)^n \left( \frac{+1}{2!} \right) = \left( O_L^n + \sum_{k=0}^{n-1} O_L^{n-1-k} O_R O_L^k + \cdots + O_R^n \right) \left( \frac{+1}{2!} \right) \quad (7)$$

This is equivalent to summing terms on the  $n$ -th row of nodes across the Tree.

Bernoulli numbers are then simply given by

$$B_n = n! S_{n-1} \quad \text{for } n \geq 2 \quad (8)$$

$$\begin{aligned} \text{eg., } B_3 &= 3! S_2 = 3! (O_L + O_R)^2 \left( \frac{+1}{2!} \right) = 3! (O_L + O_R)(O_L + O_R) \left( \frac{+1}{2!} \right) \\ &= 3! (O_L O_L + O_L O_R + O_R O_L + O_R O_R) \left( \frac{+1}{2!} \right) \\ &= 3! \left( \frac{+1}{4!} + \frac{-1}{2!3!} + \frac{-1}{3!2!} + \frac{+1}{2!2!2!} \right) = 0 \end{aligned}$$

By observation, this Sum-across-the-Tree representation of  $S_n$  is exactly equivalent to the following determinant known to generate  $B_n$ ,

$$S_n = (-1)^n \begin{vmatrix} \frac{1}{2!} & 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{1}{(n-2)!} & \ddots & \ddots & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 \\ \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \ddots & \ddots & \frac{1}{3!} & \frac{1}{2!} & 1 \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \ddots & \ddots & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} \quad (9)$$

### 3 Analytic Continuations

#### 3.1 Analytic Continuation of Operator

First, we introduce the idea of analytic continuing the action of an operator following [9]. We are used to thinking of an operator acting once, twice, three times, and so on. However, an operator acting integer times can be analytic continued to an operator acting complex times by making the following observation:

A generic operator  $A$  acting complex  $s$ -times can be formally expanded into a series as

$$\begin{aligned} A^s &= \left( w \mathbf{1} - [w \mathbf{1} - A] \right)^s = w^s \left( 1 - \left[ 1 - \frac{1}{w} A \right] \right)^s \\ &= w^s \left( \mathbf{1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left[ \prod_{k=0}^{n-1} (s-k) \right] \left[ 1 - \frac{1}{w} A \right]^n \right) \\ &= w^s \left( \mathbf{1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left[ \prod_{k=0}^{n-1} (s-k) \right] \left[ 1 + \sum_{m=1}^n \left( \frac{-1}{w} \right)^m \binom{n}{m} A^m \right] \right) \quad (10) \end{aligned}$$

where  $s \in \mathbb{C}$ ,  $w \in \mathbb{R}$ , and  $\mathbf{1}$  is the identity operator.

The region of convergence in  $s$  and the rate of convergence of the series will in general be dependent on operator  $A$ , parameter  $w$ , and the operand on which  $A$  acts.

#### 3.2 Analytic Continuation of the Tree-Generating Operator

Just as in (10), the Tree-generating operator  $(O_L + O_R)$  acting  $(s-1)$  times on  $\left(\frac{+1}{2!}\right)$  can be similarly expanded as

$$\begin{aligned} (O_L + O_R)^{s-1} \left( \frac{+1}{2!} \right) &= w^s \left( \mathbf{1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left[ \prod_{k=1}^n (s-k) \right] \left[ 1 + \sum_{m=1}^n \left( \frac{-1}{w} \right)^m \binom{n}{m} (O_L + O_R)^m \right] \right) \left( \frac{+1}{2!} \right) \\ &= w^s \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left[ \prod_{k=1}^n (s-k) \right] \left[ \frac{1}{2} + \sum_{m=1}^n \left( \frac{-1}{w} \right)^m \binom{n}{m} \frac{B_{m+1}}{(m+1)!} \right] \right) \quad (11) \end{aligned}$$

which converges for  $\text{Re}(s) > (1/w)$  where  $s \in \mathbb{C}$ ,  $w \in \mathbb{R}$ ,  $w > 0$ .

#### 3.3 Analytic Continuation of Bernoulli Numbers

$$B_n = n! (O_L + O_R)^{n-1} \left( \frac{+1}{2!} \right) = \Gamma(1+n) (O_L + O_R)^{n-1} \left( \frac{+1}{2!} \right) \quad (12)$$

Now that we can analytic continue the Tree-generating operator  $(O_L + O_R)$  with (11), if we do so, we turn the sequence of Bernoulli numbers  $B_n$  into their analytic continuation — a function

$$\begin{aligned} B(s) &= \Gamma(1+s) (O_L + O_R)^{s-1} \left( \frac{+1}{2!} \right) \quad (13) \\ &= w^s \Gamma(1+s) \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left[ \prod_{k=1}^n (s-k) \right] \left[ \frac{1}{2} + \sum_{m=1}^n \left( \frac{-1}{w} \right)^m \binom{n}{m} \frac{B_{m+1}}{(m+1)!} \right] \right) \end{aligned}$$

which converges for  $\text{Re}(s) > (1/w)$ , real  $w > 0$ .

So effectively, by the method of analytic continuation of operator, we have now obtained the function  $B(s)$  as the analytic continuation of Bernoulli numbers.

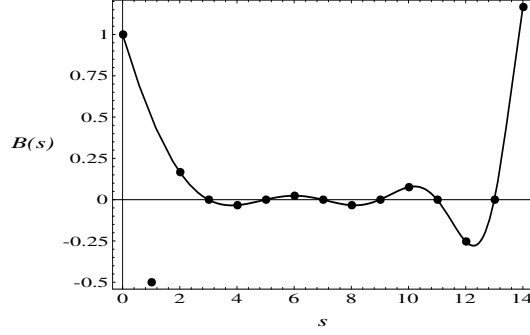


Figure 2: The curve  $B(s)$  runs through the points of all  $B_n$  except  $B_1$ .

All the Bernoulli numbers  $B_n$  agree with  $B(n)$ , the analytic continuation of Bernoulli numbers evaluated at  $n$ ,

$$B(n) = B_n \quad \text{for } n \geq 2 \quad (14)$$

$$\text{except } B(1) = \frac{1}{2} \quad \text{but} \quad B_1 = -\frac{1}{2} \quad (15)$$

## 4 Missing Signs in the Definition of $B_n$

Looking back at (1) to (4), we can see that the sign convention of  $B_1$  was actually arbitrary. (15) suggests that consistent definition of Bernoulli numbers *should really have been*

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} (-1)^n \frac{B_n}{n!} z^n, \quad |z| < 2\pi, \quad n = 1, 2, 3, \dots \in \mathbb{Z}^+ \quad (16)$$

or

$$B_n = \frac{(-1)^{n+1}}{n+1} \sum_{k=0}^{n-1} (-1)^k \binom{n+1}{k} B_k, \quad B_0 = 1 \quad (17)$$

which only changes the sign in the conventional definition of the only non-zero odd Bernoulli numbers,  $B_1$ , from  $B_1 = -1/2$  to  $B_1 = B(1) = 1/2$ .

So here's my little appeal to the Mathematics, Physics, Engineering, and Computing communities to introduce the missing signs into the sum in the definition of Bernoulli numbers as in (16) and (17) because the analytic continuation of Bernoulli numbers fixes the arbitrariness of the sign convention of  $B_1$ .

## 5 A New Formula for the Riemann Zeta Function

Bernoulli numbers are related to the Riemann zeta function as [2] [3]

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \quad (18)$$

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!} B_{2n} \quad (19)$$

for  $n = 0, 1, 2, \dots \in \mathbb{Z}_0^+$ .

From the above analytic continuation of Bernoulli numbers,

$$\begin{aligned} B_n &\mapsto B(s) \\ \left( \zeta(-n) = -\frac{B_{n+1}}{n+1} \right) &\mapsto \left( \zeta(-s) = -\frac{B(s+1)}{s+1} \right) \\ \Rightarrow \zeta(1-s) &= -\frac{B(s)}{s} \end{aligned} \quad (20)$$

Replacing  $B(s)$  in (20) with the series in (13) and noting that  $\Gamma(1+s)/s = \Gamma(s)$  gives

$$\zeta(1-s) = -w^s \Gamma(s) \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left[ \prod_{k=1}^n (s-k) \right] \left[ \frac{1}{2} + \sum_{m=1}^n \left( \frac{-1}{w} \right)^m \binom{n}{m} \frac{B_{m+1}}{(m+1)!} \right] \right) \quad (21)$$

which converges for  $\text{Re}(1-s) < 1 - (1/w)$ , real  $w > 0$ .

The functional equation of the Riemann zeta function relates  $\zeta(1-s)$  to  $\zeta(s)$  as

$$\zeta(1-s) = 2(2\pi)^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s) \quad (22)$$

Applying this relation to (21) yields

$$\cos\left(\frac{\pi \hat{s}}{2}\right) \zeta(s) = -\frac{(2\pi w)^s}{2} \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left[ \prod_{k=1}^n (s-k) \right] \left[ \frac{1}{2} + \sum_{m=1}^n \left( \frac{-1}{w} \right)^m \binom{n}{m} \frac{B_{m+1}}{(m+1)!} \right] \right) \quad (23)$$

or in the limiting form

$$\begin{aligned} \zeta(s) &= -\frac{(2\pi w)^s}{2} \lim_{\hat{s} \rightarrow s} \left\{ \frac{\left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left[ \prod_{k=1}^n (\hat{s}-k) \right] \left[ \frac{1}{2} + \sum_{m=1}^n \left( \frac{-1}{w} \right)^m \binom{n}{m} \frac{B_{m+1}}{(m+1)!} \right] \right)}{\cos\left(\frac{\pi \hat{s}}{2}\right)} \right\} \\ &= -\frac{(2\pi w)^s}{2} \lim_{\hat{s} \rightarrow s} \left\{ \frac{\left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left[ \prod_{k=1}^n (\hat{s}-k) \right] \left[ \frac{1}{2} - \sum_{m=1}^n \left( \frac{-1}{w} \right)^m \binom{n}{m} \frac{\zeta(-m)}{m!} \right] \right)}{\cos\left(\frac{\pi \hat{s}}{2}\right)} \right\} \end{aligned} \quad (24)$$

a nested sum of the Riemann zeta function itself evaluated at negative integers, which converges for  $\text{Re}(s) > (1/w)$ , real  $w > 0$ , and the limit only needs to be taken when  $s = 1, 3, 5, \dots \in \mathbb{Z}_{odd}^+$ , the set of positive odd integers, for which the denominator  $\cos\left(\frac{\pi s}{2}\right) = 0$ .

This is consistent with

$$\begin{aligned} \left( \zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!} B_{2n} \right) &\mapsto \left( \zeta(2s) = -\frac{(2\pi)^{2s}}{2\Gamma(1+2s)} \lim_{\hat{s} \rightarrow s} \left\{ \frac{B(2\hat{s})}{\cos(\pi \hat{s})} \right\} \right) \\ \Rightarrow \zeta(s) &= -\frac{(2\pi)^s}{2\Gamma(1+s)} \lim_{\hat{s} \rightarrow s} \left\{ \frac{B(\hat{s})}{\cos\left(\frac{\pi \hat{s}}{2}\right)} \right\} \end{aligned} \quad (25)$$

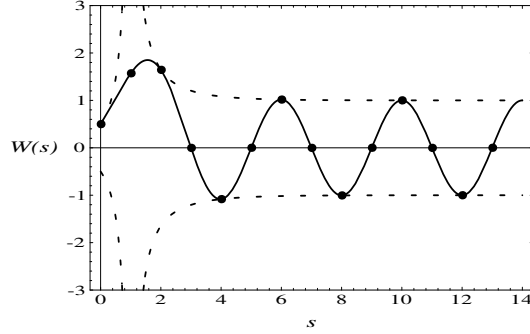


Figure 3:  $W(s) = \frac{(2\pi)^s B(s)}{2\Gamma(1+s)} = -\cos\left(\frac{\pi s}{2}\right) \zeta(s)$ .  $W(s)$  is bounded by the envelopes  $\pm |\zeta(s)|$  shown as dashed curves.

Since both  $B(k)$  and  $\cos\left(\frac{\pi k}{2}\right) = 0$  for odd integer  $k \in \mathbb{Z}_{odd}^+ \setminus \{1\}$ , L'Hôpital rule can be applied to the limit giving

$$\lim_{\hat{s} \rightarrow s} \left\{ \frac{B(\hat{s})}{\cos\left(\frac{\pi \hat{s}}{2}\right)} \right\} = \begin{cases} \frac{B(s)}{\cos\left(\frac{\pi s}{2}\right)} & , \quad s \notin \mathbb{Z}_{odd}^+ \\ \infty & , \quad s = 1 \\ (-1)^{(s+1)/2} \frac{2B'(s)}{\pi} & , \quad s \in \mathbb{Z}_{odd}^+ \setminus \{1\} \end{cases} \quad (26)$$

where the prime denotes differentiation and so  $B'(s)$  is the derivative of the function  $B(s)$ .

Therefore we have now found the apparently missing 'odd' expression 'dual' to the 'even' expression (19).

$$\text{(even):} \quad \zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{(2n)!} \frac{B_{2n}}{2} \quad (27)$$

$$\text{(odd):} \quad \zeta(2n+1) = (-1)^n \frac{(2\pi)^{2n+1}}{(2n+1)!} \frac{B'(2n+1)}{\pi} \quad (28)$$

for  $n = 1, 2, 3, \dots \in \mathbb{Z}^+$ .

In fact, we can express  $B'(s)$  in terms of  $\zeta'(s)$ , the derivative of  $\zeta(s)$ .

$$\begin{aligned} B(s) &= -s \zeta(1-s) \\ B'(s) &= -\zeta(1-s) + s \zeta'(1-s) \\ B'(2n+1) &= (2n+1) \zeta'(-2n) \end{aligned} \quad (29)$$

as  $\zeta(-2n) = 0$

$$\Rightarrow \quad \zeta(2n+1) = (-1)^n \frac{(2\pi)^{2n+1}}{(2n)! \pi} \zeta'(-2n) \quad \text{for } n \in \mathbb{Z}^+. \quad (30)$$

This is just the differential of the functional equation and so verifies the consistency of  $B(s)$  and  $B'(s)$  with  $B_n$  and  $\zeta(s)$ .

## 6 Other Undiscovered Half of Bernoulli Numbers

From the relation (20), we can define the other analytic continued half of Bernoulli Numbers

$$\begin{aligned} B(s) &= -s \zeta(1-s), \quad B(-s) = s \zeta(s+1) \\ \Rightarrow B_{-n} &= B(-n) = n \zeta(n+1), \quad n \in \mathbb{Z}^+ \end{aligned} \quad (31)$$

Since  $\zeta(n+1) \rightarrow 1$  as  $n \rightarrow \infty$ ,  $B_{-n} \sim n$  asymptotically for  $(-n) \ll -1$ .

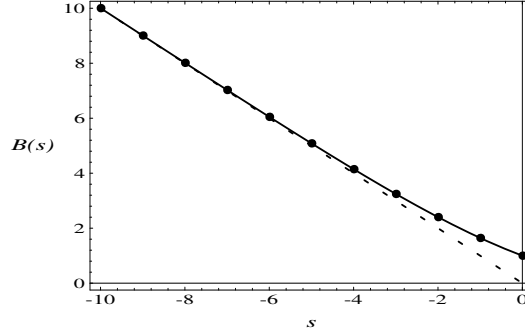


Figure 4: The curve  $B(s)$  runs through the points  $B_{-n}$  and grows  $\sim n$  asymptotically as  $(-n) \rightarrow -\infty$ .

## 7 More Related Analytic Continuations

### 7.1 Analytic Continuation of Bernoulli Polynomials

The conventional definition [2] [3] of Bernoulli polynomials  $B_n(x)$  also has an arbitrariness in the sign convention. For consistency with the redefinition of  $B_n = B(n)$  in (16) and (17), Bernoulli polynomials should be analogously *redefined* as

$$\begin{aligned} \frac{z e^{xz}}{e^z - 1} &= \left( \sum_{k=0}^{\infty} (-1)^k B_k \frac{z^k}{k!} \right) \left( \sum_{m=0}^{\infty} \frac{(xz)^m}{m!} \right) = \sum_{n=0}^{\infty} (-1)^n B_n(x) \frac{z^n}{n!} \\ \Rightarrow B_n(x) &= \sum_{k=0}^n (-1)^{(n+k)} \binom{n}{k} B_k x^{n-k} \end{aligned} \quad (32)$$

The analytic continuation can be then obtained as

$$\begin{aligned} n &\mapsto s \in \mathbb{R}, s \geq 1, \quad x \mapsto w \in \mathbb{C} \\ B_k &\mapsto B(k + (s - [s])) = -(k + (s - [s])) \zeta(1 - (k + (s - [s]))) \\ \binom{n}{k} &\mapsto \frac{\Gamma(1+s)}{\Gamma(1+k + (s - [s])) \Gamma(1 + [s] - k)} \\ \Rightarrow B_n(x) &\mapsto B(s, w) = \sum_{k=-1}^{[s]} \frac{(1)^{n+k} \Gamma(1+s) B(k + (s - [s])) w^{[s]-k}}{\Gamma(1+k + (s - [s])) \Gamma(1 + [s] - k)} \end{aligned}$$



$$\sum_{k=0}^{[s]+1} \frac{(1)^{n+k+1} \Gamma(1+s) B((k-1) + (s-[s])) w^{[s]+1-k}}{\Gamma(k + (s-[s])) \Gamma(2 + [s] - k)} \quad (33)$$

where  $[s]$  gives the integer part of  $s$ , and so  $(s-[s])$  gives the fractional part,

$$\begin{aligned} \text{eg., } B_2(x) &= B(2, x) = 1/6 - x + x^2 \approx 0.16667 - x + x^2 \\ B(2.01, x) &\approx 0.16420 - 0.99660x + 1.00920x^2 - 0.00554x^3 \\ B(2.99, x) &\approx 0.00092 - 0.50576x + 1.50001x^2 - 0.98744x^3 \\ B_3(x) &= B(3, x) = -0.5x + 1.5x^2 - x^3 \end{aligned}$$

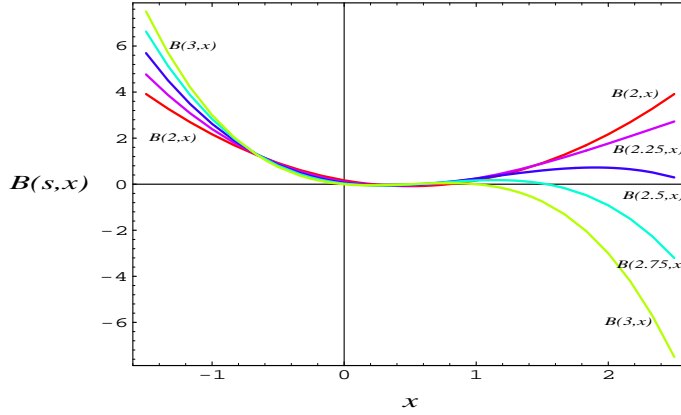


Figure 5: Deformation of the curve of  $B_2(x) = B(2, x)$  into the curve of  $B_3(x) = B(3, x)$  via the real analytic continuation  $B(s, x)$ ,  $2 \leq s \leq 3$ ,  $x \in \mathbb{R}$ .

## 7.2 Analytic Continuation of Euler Numbers and Euler Polynomials

Euler numbers  $E_n$  and Euler polynomials  $E_n(x)$  are defined [2] [3] as

$$\text{sech } z = \frac{2 e^z}{e^{2z} + 1} = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!}, \quad |z| < 2\pi, \quad n \in \mathbb{Z}^+; \quad E_n = 0 \text{ for odd } n \quad (34)$$

$$\begin{aligned} \frac{2 e^{xz}}{e^{2z} + 1} &= \frac{2 e^{z/2} e^{z(x-1/2)}}{(e^z + 1)^{-1}} = \left( \sum_{k=0}^{\infty} E_k \frac{z^k}{2^k k!} \right) \left( \sum_{m=0}^{\infty} \frac{(x - \frac{1}{2})^m z^m}{m!} \right) = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} \\ \Rightarrow E_n(x) &= \sum_{k=0}^n \binom{n}{k} 2^{-k} \left(x - \frac{1}{2}\right)^{n-k} E_k \end{aligned} \quad (35)$$

and are related to Bernoulli polynomials as

$$E_n(x) = \frac{2}{n+1} \left( B_{n+1}(x) - 2^{n+1} B_{n+1}\left(\frac{x}{2}\right) \right) \quad (36)$$

$$E_n = 2^n E_n\left(\frac{1}{2}\right) = \frac{2}{n+1} \left( B_{n+1}\left(\frac{1}{2}\right) - 2^{n+1} B_{n+1}\left(\frac{1}{4}\right) \right) \quad (37)$$

Their analytic continuation then follows straightforwardly from the analytic continuation of Bernoulli polynomials

$$B_n(x) \mapsto B(s, w), \quad E_n \mapsto E(s), \quad E_n(x) \mapsto E(s, w)$$

where  $n \mapsto s \in \mathbb{R}$ ,  $s \geq 1$ ,  $x \mapsto w \in \mathbb{C}$ .

## 8 Beautiful zeros of Bernoulli Polynomials

### 8.1 Distribution and Structure of the zeros

Zeros of Bernoulli polynomials are solutions of  $B_n(w) = 0$ ,  $w \in \mathbb{C}$ .

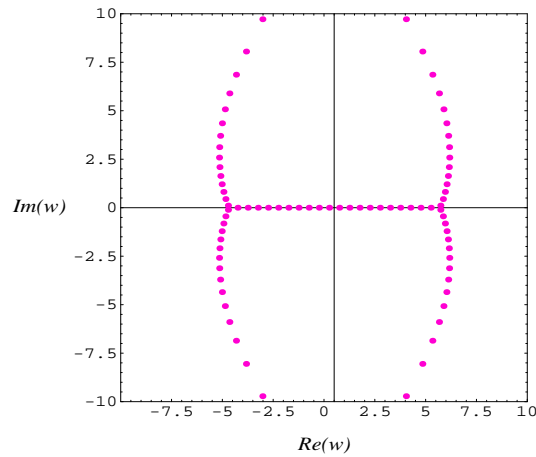


Figure 6: Zeros of Bernoulli polynomials  $B_{80}(w)$ .

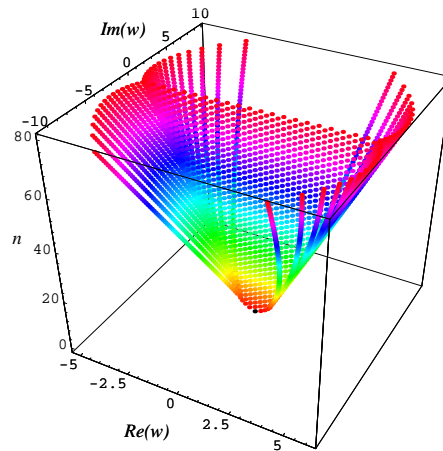


Figure 7: Stacks of zeros of Bernoulli polynomials  $B_n(w)$  form a 3-D structure.

The real zeros, except the outermost pair in general, are almost regularly spaced, while the complex zeros lie on arcs symmetrical about  $\text{Re}(w) = 1/2$ .

## 8.2 Observations, Theorem, Conjectures, and Open Problems

### 1. *Symmetries*

Prove that  $B_n(w)$ ,  $w \in \mathbb{C}$ , has  $\text{Re}(w) = 1/2$  reflection symmetry in addition to the usual  $\text{Im}(w) = 0$  reflection symmetry analytic complex functions. The obvious corollary is that the zeros of  $B_n(w)$  will also inherit these symmetries.

$$\text{If } B_n(w_0) = 0, \text{ then } B_n(1 - w_0) = 0 = B_n(w_0^\dagger) = B_n(1 - w_0^\dagger) \quad (38)$$

where  $^\dagger$  denotes complex conjugation.

### 2. *Non-degenerate zeros*

Prove that  $B_n(w) = 0$  has  $n$  distinct solutions, ie., all the zeros are non-degenerate.

### 3. *Central zero Theorem*

If  $B_{2m+1}$  has  $\text{Re}(w) = 1/2$  and  $\text{Im}(w) = 0$  reflection symmetries, and  $(2m+1)$  non-degenerate zeros, then  $(2m)$  of the distinct zeros will satisfy (38). If the remaining one zero is to satisfy (38) too, it must reflect into itself, and therefore it must lie at  $1/2$ , the center of the structure of the zeros, ie.,

$$B_n(1/2) = 0 \quad \forall \text{ odd } n \quad (39)$$

### 4. *Counting of real and complex zeros*

Prove that the number of complex zeros  $\mathbf{z}_{\mathbb{C}(n)}$  of  $B_n(w)$  lying on the 4 sets of arcs off the real plane,  $\text{Im}(w) \neq 0$ , is

$$\mathbf{z}_{\mathbb{C}(n)} = 4 \left\lfloor \frac{n - \delta}{5} \right\rfloor, \text{ where } \delta = \left\lfloor \frac{n + 30}{21} \right\rfloor \quad (40)$$

and  $\lfloor \cdot \rfloor$  denotes taking the integer part. The factor 4 comes from the above 2 reflection symmetries.

Since  $n$  is the degree of the polynomial  $B_n(w)$ , the number of real zeros  $\mathbf{z}_{\mathbb{R}(n)}$  lying on the real plane  $\text{Im}(w) = 0$  is then  $\mathbf{z}_{\mathbb{R}(n)} = (n - \mathbf{z}_{\mathbb{C}(n)})$ .

See Appendix for tabulated values of  $\mathbf{z}_{\mathbb{R}(n)}$  and  $\mathbf{z}_{\mathbb{C}(n)}$ .

### 5. *Asymptotic Lattice of real zeros*

It is known that from (7.1),

$$\begin{aligned} \sum_{n=0}^{\infty} (B_n(w+1) - B_n(w)) \frac{z^n}{n!} &= z e^{wz} = \sum_{n=1}^{\infty} \frac{w^{n-1} z^n}{(n-1)!} \\ \Rightarrow B_n(w+1) - B_n(w) &= w^{n-1} \\ \Rightarrow B_n(1) = B_n(0) = B_n &= 0 \quad \forall \text{ odd } n \geq 3 \end{aligned}$$

Show that all the real zeros of  $B_n(w)$  except the outermost pair in general, are approximately regularly spaced at the staggered lattice points

$$w = \begin{cases} 0, \pm 1/2, \pm 1, \pm 3/2, \dots & \in \mathbb{Z}/2 & \text{for odd } n \\ \pm 1/4, \pm 3/4, \pm 5/4, \dots & \in \mathbb{Z}/2 + 1/4 & \text{for even } n \end{cases} \quad (41)$$

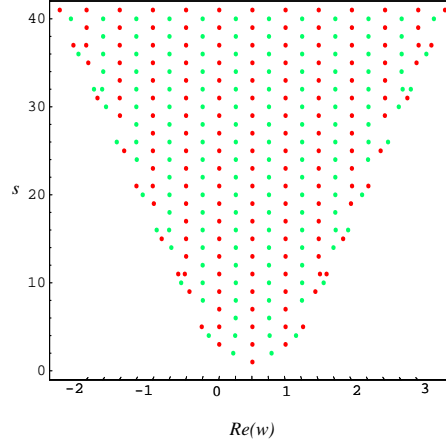


Figure 8: Inner real zeros converge to a staggered Lattice structure.

and becomes increasingly located exactly at these lattice points as  $n \rightarrow \infty$ .  
See Appendix for tabulated solutions of  $B_n(w) = 0$ .

#### 6. Relation between zeros of Bernoulli and Euler polynomials

Choose any zero of Bernoulli polynomial  $B_n(w)$  and denote it as  $w_{B(n)}$ ,  
ie.  $B_n(w_{B(n)}) = 0$ .

Prove that

$$\text{as } n \rightarrow \infty, \quad E_n(2(w_{B(n)} - 1/2) + 1/2) = E_n(2w_{B(n)} - 1/2) = 0 \quad (42)$$

ie., the structure of the zeros of Euler polynomials resembles the structure of the zeros of Bernoulli polynomials but doubled in size in the limit the degree of the polynomials  $n \rightarrow \infty$ . Both structures are centered at  $w = 1/2$ .

#### 7. Bounding Envelopes and Trajectories of complex zeros

Find the equation of envelope curves bounding the real zeros lying on the plane, and the equation of a trajectory curve running through the complex zeros on any one of the arcs.

### 8.3 Dynamics of the zeros from Analytic Continuation

Bernoulli polynomial  $B_n(w)$  is a polynomial of degree  $n$ . Thus,  $B_n(w)$  has  $n$  zeros and  $B_{n+1}(w)$  has  $(n+1)$  zeros. When discrete  $n$  is analytic continued to continuous parameter  $s$ , it naturally leads to the question:

How does  $B(s, w)$ , the analytic continuation of  $B_n(w)$ , pick up an additional zero as  $s$  increases continuously by one?

This introduces the exciting concept of the dynamics of the zeros of analytic continued polynomials — the idea of looking at how the zeros move about in the  $w$  complex plane as we vary the parameter  $s$ .

Continuity shows that the additional zero simply “flows in from infinity”.

To have a physical picture of the motion of the zeros in the complex  $w$  plane, imagine that each time as  $s$  increases gradually and continuously by one, an additional real zero flies in from positive infinity along the real positive axis, gradually slowing down as if “it is flying through a viscous medium”.

For  $s < 5$ , the additional zero simply joins onto the almost regularly spaced array of the real zeros streaming slowly towards the negative real direction. The array of zeros continue to drift freely until one by one they hit a bounding envelope which grows in size with  $s$ .

As  $s$  approaches every integer  $> 5$ , an interesting phenomenon occurs: A pair of real zeros may meet and become a doubly degenerate real zero at a point, and then bifurcate into a pair of complex zeros conjugate to each other. Thus, the pair of real zeros appears to “collide head-on and scatter perpendicularly” into a pair of complex zeros.

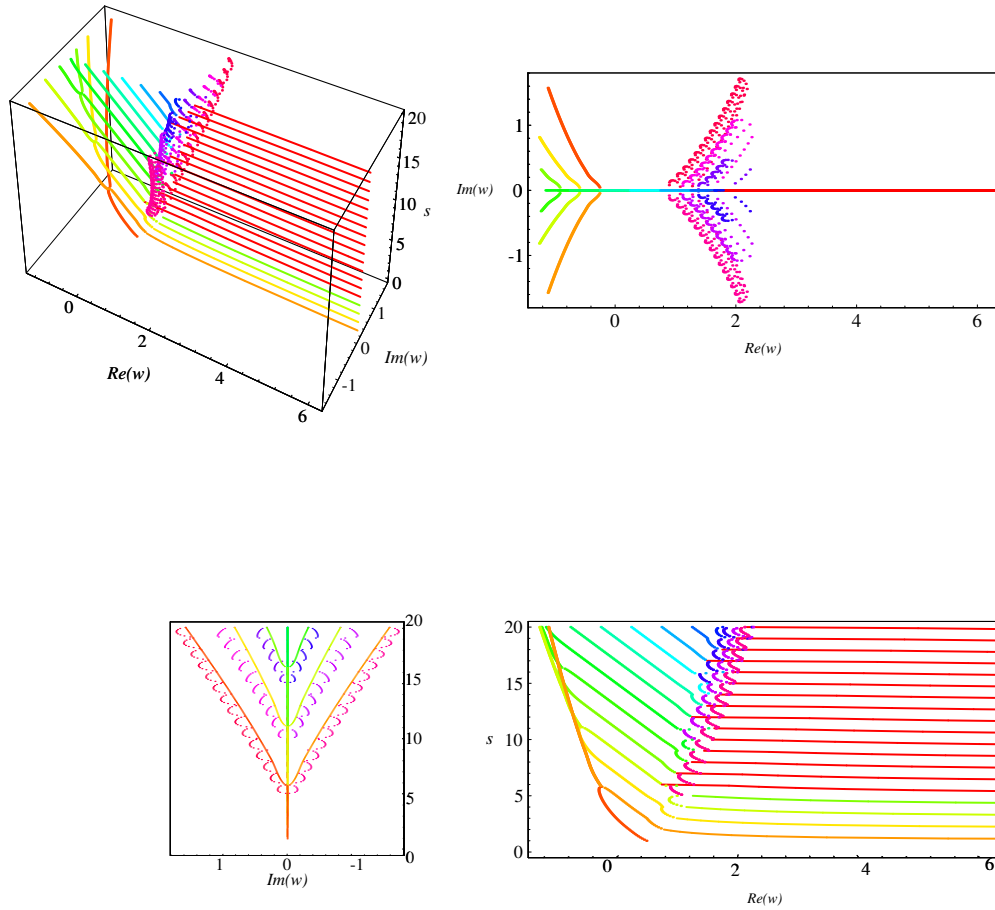


Figure 9: The flow of zeros of  $B(s, w)$  forms a complex 3-D structure.

3 fundamental kinds of scattering can be observed:

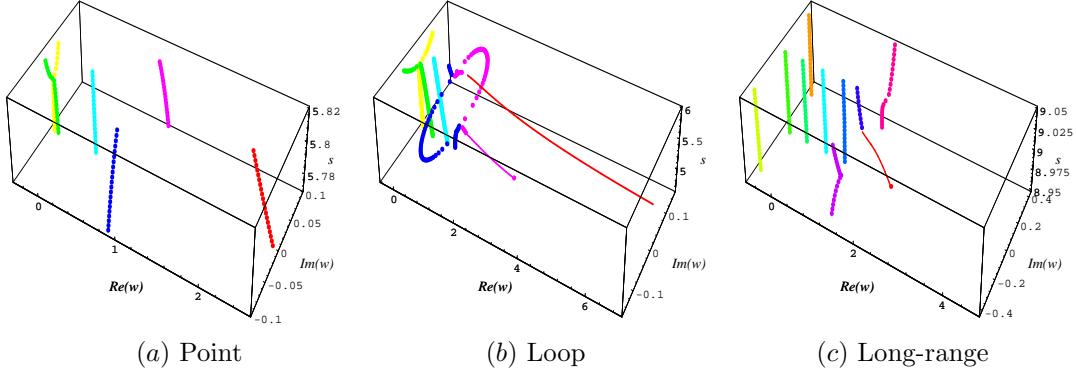


Figure 10: 3 fundamental kinds of scattering.

- Point scattering:  
A pair of real zeros scatter at a point into a pair of complex zeros which head away from each other indefinitely.
- Loop scattering:  
The same as point scattering but the pair of complex zeros loops back to recombine into degenerate real zeros within unit interval in  $s$  and then scatter back into a pair of real zeros, much like the picture of pair production and annihilation of virtual particles.
- Long-range sideways scattering:  
The additional zero that flies in appears as if “it is carrying with it a line front of shockwave” that stretches parallelly to the  $\text{Im}(w)$  axis. When the “shockwave” meets a pair of complex zeros that are looping back, the pair gets deflected away from each other momentarily before looping back again, while the additional zero gets perturbed and slows down discontinuously.

The whole complex structure can then be reduced to simple combinations of these 3 kinds of scattering.

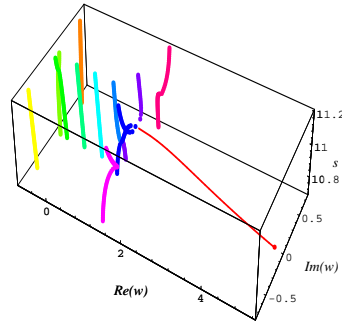


Figure 11: A composite of Point, Loop and Long-range scatterings.

The movies (animated gifs) showing the motion of zeros in the complex  $w$  plane at slices of  $s$  can be downloaded from Internet at [11] and viewed with Netscape or Internet Explorer WWW browser.

## 8.4 Open Challenges in Generalization

1. Generalize the above results of  $B(s, w)$  consistently from  $s \in \mathbb{R}$ ,  $s \geq 1$  to  $s \in \mathbb{C}$ .
2. Derive a set of expressions which give the values of  $s$  where the point, loop and long-range scatterings occur.

## 9 Conclusion

Bernoulli numbers and polynomials appear in many areas. In particular, if we assume the Riemann Hypothesis, when  $s \in \mathbb{C}$ , (24) should converge to zero only on the line  $s = 1/2 \pm it$ . This remains to be proved. More of this aspect is analysed in [10].

To further self-explore these fascinating properties, feel free to download and adapt the executable *Mathematica* codes from [12]. Have fun!

In the meantime, it would be interesting to imagine what Jakob Bernoulli and Euler would say on these analytic continuations of their numbers and polynomials.

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## Appendix:

### Number of real and complex zeros of $B_n(w)$

$B_n(x)$ degree $n$	real zeros $\mathbf{z}_{\mathbf{R}(n)}$	complex zeros $\mathbf{z}_{\mathbf{C}(n)}$
1	1	0
2	2	0
3	3	0
4	4	0
5	5	0
6	2	4
7	3	4
8	4	4
9	5	4
10	6	4
11	7	4
12	4	8
13	5	8
14	6	8
15	7	8
16	8	8
17	5	12
18	6	12
19	7	12
20	8	12
21	9	12
22	6	16
23	7	16
24	8	16
25	9	16
26	10	16

$B_n(x)$ degree $n$	real zeros $\mathbf{z}_{\mathbf{R}(n)}$	complex zeros $\mathbf{z}_{\mathbf{C}(n)}$
27	7	20
28	8	20
29	9	20
30	10	20
31	11	20
32	12	20
33	9	24
34	10	24
35	11	24
36	12	24
37	13	24
38	10	28
39	11	28
40	12	28
41	13	28
42	14	28
43	11	32
44	12	32
45	13	32
46	14	32
47	15	32
48	12	36
49	13	36
50	14	36
51	15	36
52	16	36
53	17	36

$B_n(x)$ degree $n$	real zeros $\mathbf{z}_{\mathbf{R}(n)}$	complex zeros $\mathbf{z}_{\mathbf{C}(n)}$
54	14	40
55	15	40
56	16	40
57	17	40
58	18	40
59	15	44
60	16	44
61	17	44
62	18	44
63	19	44
64	16	48
65	17	48
66	18	48
67	19	48
68	20	48
69	17	52
70	18	52
71	19	52
72	20	52
73	21	52
74	22	52
75	19	56
76	20	56
77	21	56
78	22	56
79	23	56
80	20	60
$\vdots$	$\vdots$	$\vdots$



### Solutions of $B_n(w) = 0$

odd $n$	$w$
1	$\left(\frac{1}{2}\right)$
3	$\left(0\right), \left(\frac{1}{2}\right), \left(1\right)$
5	$\left(0\right), \left(\frac{1}{2}\right), \left(1\right),$ $\left(\frac{1}{2} - \frac{\sqrt{7/3}}{2}\right) \approx -0.26376, \left(\frac{1}{2} + \frac{\sqrt{7/3}}{2}\right) \approx 1.26376$
7	$\left(0\right), \left(\frac{1}{2}\right), \left(1\right),$ $\left(\frac{1}{2} - \frac{\sqrt{9+2\sqrt{3}i}}{2\sqrt{3}}\right) \approx -0.38137 - 0.16376i,$ $\left(\frac{1}{2} - \frac{\sqrt{9-2\sqrt{3}i}}{2\sqrt{3}}\right) \approx -0.38137 + 0.16376i,$ $\left(\frac{1}{2} + \frac{\sqrt{9-2\sqrt{3}i}}{2\sqrt{3}}\right) \approx 1.38137 - 0.16376i,$ $\left(\frac{1}{2} + \frac{\sqrt{9+2\sqrt{3}i}}{2\sqrt{3}}\right) \approx 1.38137 + 0.16376i$

even $n$	$w$
2	$\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) \approx 0.21133, \left(\frac{1}{2} + \frac{1}{2\sqrt{3}}\right) \approx 0.78868$
4	$\left(\frac{1}{2} - \frac{\sqrt{15+2\sqrt{30}}}{2\sqrt{15}}\right) \approx -0.15770,$ $\left(\frac{1}{2} - \frac{\sqrt{15-2\sqrt{30}}}{2\sqrt{15}}\right) \approx 0.24034,$ $\left(\frac{1}{2} + \frac{\sqrt{15-2\sqrt{30}}}{2\sqrt{15}}\right) \approx 0.75967,$ $\left(\frac{1}{2} + \frac{\sqrt{15+2\sqrt{30}}}{2\sqrt{15}}\right) \approx 1.15770$